Note to the curator of the Göttingen mathematical models collection

Dear colleague,

During our guided tour in your magnificent collection on Monday, November 28, 2018, we encountered some 'unknown' gypsum models (Figure 1). In my opinion, these are related to Saint-Venant's research[1] on torsion of prisms. Furthermore, it seems likely that the models were made by Saint-Venant.

Figure 1: Unlabelled models in Göttingen collection

1. Saint-Venant's models in two and three dimensions

If a rod with cylindrical section undergoes torsion, the cross sections will remain flat. The research of Saint-Venant on the torsion of prisms showed how the internal body is subjected to stress resulting in distorted cross sections, if cross sections are not circular, as in prism. In Isaac Todhunter's *The history of elasticity*[2], the research of Saint-Venant is discussed in detail and the pictures are 2D versions of the gypsum models in your collection. In Figures 2 and 3 the results are shown for prisms with square, triangular and rectangular cross sections. The plaster models show the real three-dimensional deformations. I could not observe all the numbers, but in your collection the square is N°595, the ellipse is 596°, the triangle is N°597 and the star with four rounded points is N°598.

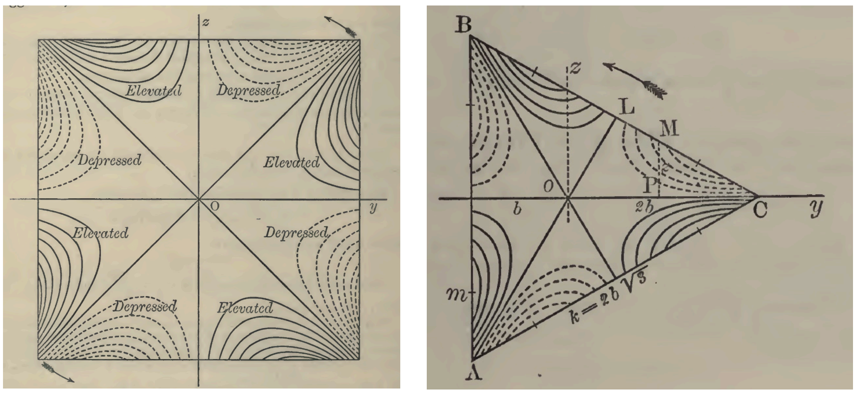

Figure 2: Torsion of prisms. Results for prism with square and triangular cross-sections. Dotted lines are depressed regions and full lines are elevated parts of the cross sections.

[1] Adhémar Jean Claude Barré de Saint-Venant (1797-1886) is a contemporary of Gabriel Lamé (1795-1860) of which some models are displayed in the collection.

[2] *The history of elasticity and of the Strength of Materials – from Galilei to the present time.* By the late Isaac Todhunter, edited and completed for the syndics of the University Press by Karl A. Pearson, Volumer II Saint-Venant to Lord Kelvin, Part 1. Cambridge at the University Press, 1893.

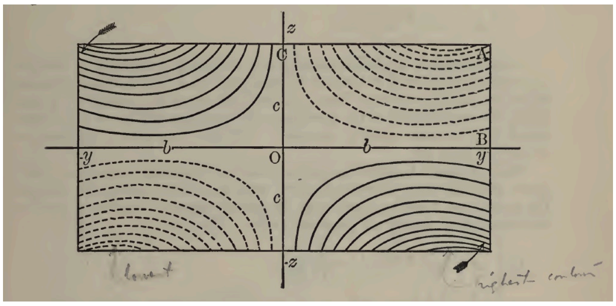

Figure 3: Torsion of prisms. Results for prisms with rectangular cross-section. Dotted lines are depressed regions and full lines are elevated parts of the cross sections.

In the collection, there were a few special shapes, which I had not seen before. However, in *The History of Elasticity* solutions for special cross sections studied by Saint-Venant, are discussed. In Figure 4 the star with four rounded points is one of the 3D models in your collection, N°598 (the model in the centre in Figure 1). It may also be the case the square with acute angle in Figure 4 is present, but I could only look through the glass and could not distinguish whether the model is a square or a square with acute angles.

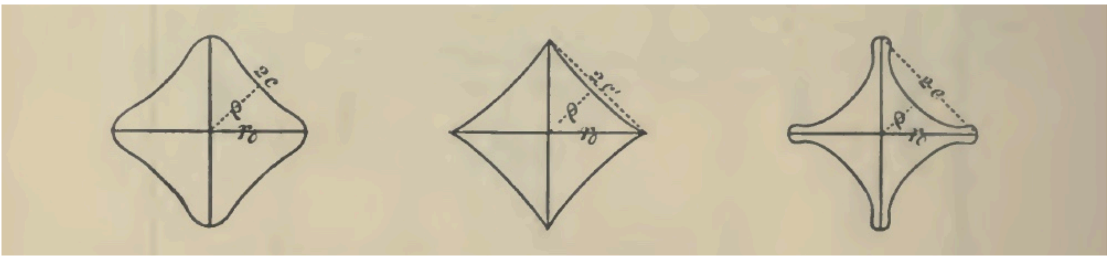

Figure 4: From left to right: square with rounded angle, square with acute angle (both fourth order), and star with four rounded points (eight order)

2. Saint-Venant as a model maker

The next question is where they were made and by whom. Interestingly, models N°846 and N°847 in your collection, displaying effects of vibrations, were actually made by Saint-Venant. The notes state that Saint-Venant provided both models in 1852 to the *Société Philomathique de Paris* (established in 1788). How they ended up in the Göttingen collection is information that may be in your possession. Otherwise, it would be interesting to investigate this further.

Given the fact that Saint-Venant was a model maker for gypsum math models, a reasonable hypothesis is that the models in Figure 1, could also be attributed directly to him. They may be related directly to N°846 and N°847, and perhaps shared a common history: made by Saint-Venant, donated to Société, then to Göttingen. An inquiry with the *Société* might clarify this, but a more direct and conclusive way would be a simple comparative chemical analysis of one of the models N°s 595-598 and one of the models N°846 and N°847. For this small samples would suffice.

3. Saint-Venant continuous to be important

The importance of the research of Saint-Venant cannot be underestimated, and his results inspired many in the past 150 years. In D'Arcy Thompson's 1917 *On Growth and Form,* a wonderful book applying many fundamental 19th century results in geometry and mathematical physics to living organisms, one can find a discussion of Saint-Venant's results in relation to the formation of horns. After making the remark that natural shapes are not accurate triangles, in the section *The shape of horns of sheep and goats* he writes:

> "*The better to illustrate this phenomenon, the nature of which is indeed obvious enough from a superficial examination of the horn, I made a plaster cast of one of the horny rings in a horn of* Ovis Ammon*, so as to get an accurate pattern of its sinuous edges and then, filling the mould up with wet clay, I modelled an anticlastic surfaces, such as to correspond as nearly as possible with the sinuous outline. Finally, after making a plaster cast of this sectional surface, I drew its contour-lines (as shewn in Figure 322), with the help of a simple form of spherometer. It will be seen that in great part this diagram is precisely similar to Saint-Venant's diagram of the cross section of a twisted trianglular prism".*

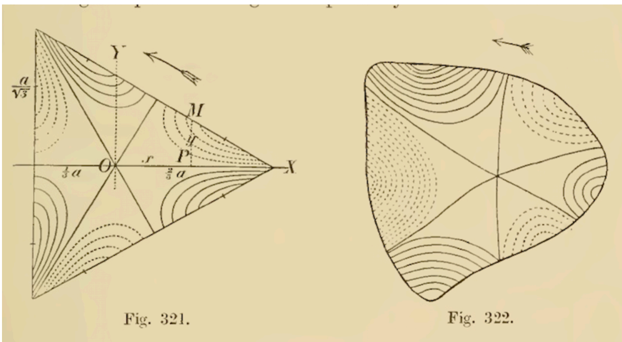

Figure 5: Figs 321 and 322 from On growth and form. Figure 321 is Saint-Venant's triangular cross section

Our own research is also directly related to Saint-Venant and to mathematical research in Göttingen. In the study of boundary value problems in the 1950 and 1960's in Moscow, Olga Oleinik observed that the solution would greatly benefit from the knowledge of the domain. My colleague Ilia Tavkhelidze from Tiblisi, then a doctoral student of Oleinik, started to generalize Möbius-Listing bodies and surfaces, and studied the result of their cutting. Hitherto, mainly the classical Möbius ribbons were studied, with cross section a line, but in our joint work the generalization allows for cross sections of different planar curves or disks, leading to surfaces or bodies respectively.

In Figure 6 one example is shown, with basic line a circle and a regular pentagon as cross section. If a cut is made from side to side and the knife follows the path until it comes back to the original position, the structure will result in 4 different complex structures. The different bodies will have cross sectional shapes according to the coloured zones (yellow, brown, gray or blue), and the number of twists of each body, will be determined by the original number of twists and the way the cut is made. In Figure 6 right only one of the bodies is shown. Obviously, in Generalized Möbius-Listing bodies with regular-polygons as cross section, the whole structure is subject to torsion and bending. The findings of Saint-Venant will be fundamental to understand this.

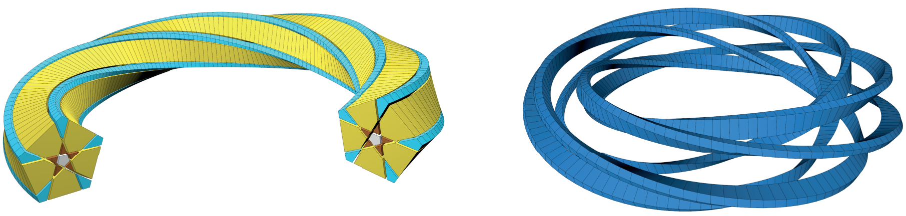

Figure 6: Generalized Möbius-Listing body and one result after cutting from side to side.

Interestingly, in the case when the polygon has an even number of sides and the knife cuts the centre of the polygon, only one body will result, displaying the Möbius phenomenon. This is directly related to non-orientable surfaces, for the first time systematically studied by Möbius and Listing, students of Pfaff and Gauss, respectively.

Most of the main research lines in mathematics initiated in Göttingen are today the driving forces of contemporary research. It is of utmost importance to know the historical backgrounds. A collection like Göttingen's Mathematical Models is, in my opinion, of extremely great value, and I hope that my letter can help solve some open questions.

Yours truly,

Johan Gielis

Antwerpen, November 29, 2018