AN ATTEMPT TO DETERMINE THE TWENTY-SEVEN LINES UPON A SURFACE OF THE THIRD ORDER, AND TO DIVIDE SUCH SURFACES INTO SPECIES IN REFERENCE TO THE REALITY OF THE LINES UPON THE SURFACE.

By DR. SCHLÄFLI, Professor of Mathematics at the University of Bern.
Translated by A. CAYLEY.

PRELIMINARY remarks. Contrary to the usual practice I would, in the case of a curve, term *singular* those points only at which Taylor's theorem fails for point coordinates, and where in consequence the tangent ceases to be linearly determined; and in like manner term *singular* those tangents for which the point of contact ceases to be linearly determined. Thus a point of inflexion is not a singular point, but the tangent at such point is a singular tangent. According to the same principle, in the case of a surface, I call singular points those only for which the tangent plane ceases to be linearly determined. I say further that a surface is general as regards order when it has no singular points, general as regards class when it has no singular tangent planes. By class I understand the number of tangent planes which pass through an arbitrary line; by singular tangent planes, the tangent planes for which the point of contact ceases to be linearly determined. By *order* of a curve in space, I mean the number of points in which the curve is intersected by an arbitrary plane, by class (as for surfaces) the number of tangent planes (planes containing a tangent of the curve) which pass through an arbitrary point. On account of their reciprocal relation to curves I guard myself from putting *developable surfaces* on a footing with proper curved surfaces, and call them therefore simply *developables* without the addition of the word surface, since they do not, like proper surfaces, arise from the double motion of a plane, but arise from the simple motion of a plane. I call indeed *degree* of a developable the number of points of intersection with an arbitrary line, but *class* the number of generating planes which pass through an arbitrary point. The representation of an algebraical curve in space requires at least two equations, that is, two surfaces passing through the curve. If these surfaces can be chosen so that their complete intersection is merely the curve in question, such curve may be termed a complete-curve (Vollcurve). But when this is not possible, and the complete intersection of any two surfaces passing through the curve consists *always* of such curve accompanied by one or more other curves, the curve in question is termed a partial-curve (Theilcurve).*

Suppose now that $f(w, x, y, z)=0$ is the homogeneous equation of an algebraical surface of the n^{th} order; w, x, y, z the coordinates of a point P of the surface, which, as the surface originally given, I will call for shortness the *basis*. Moreover let $D=w'\frac{d}{dw}+x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}$ represent a linear derivation symbol, in which the elements w', x', y', z' denote the coordinates of a point in space, which may be designated by the same letter D: the derivation symbol may be called for shortness the symbol of the point to which it relates. The system $f=0$, $Df=0$ expresses that the point D is situate in the tangent plane to the surface at the point D. This plane cuts the basis in a curve ($f=0$, $Df=0$, $D''f=0$) which has the point of contact as a double point; I will call the curve simply the contact section (Berührungschnitt). Since P is an arbitrary point upon the surface, there are in the contact section two disposable elements; when therefore we add the condition that the curve has a second double point, there remains but one disposable element; and if we assume that there are three double points in all, the plane becomes determinate. In other words, to a general (as regards order) algebraical surface of an order higher than the second, there belongs a developable, the generating planes of which touch the surface in two points. Among these generating planes there are found a determinate number of planes touching the surface in three points. The developable may be termed the doubly circumscribed developable,† the planes the triple tangent planes of the surface. The problem which next presents itself is to determine the curve along which the surface is touched by the doubly circumscribed developable.

* The names Vollcurve and Theilcurve belong to Steiner.

† (Note by the Translator). This is the developable which I have called the node-couple developable; and further on, the osculation curve is that frequently called the parabolic curve and which I have termed the spinode curve; the osculating circumscribed developable is what I have termed the spinode developable, and the self-touching double points what I have termed tacnodes. See my paper "On the Singularities of Surfaces," *Cambridge and Dublin Mathematical Journal*, t. VII. p. 166.

*Suppose as before that (w, x, y, z) are the coordinates of a point P of the basis $f=0$ and moreover that (w', x', y', z'), (w'', x'', y'', z'') are the coordinates of two points lying in the corresponding tangent plane, D', D'' their symbols in respect of P, so that $D'f=0$, $D''f=0$. If then ψ, χ, ω are three new variables, and $\psi P+\chi D'+\omega D''$ denotes a point common to the tangent plane and the basis (i.e. if $\psi w+\chi w'+\omega w''$, $\psi x+\chi x'+\omega x''$, &c. are the coordinates of the point in question) then

$$F(\psi, \chi, \omega)=\tfrac{1}{2}\psi^{n-2}(\chi D'+\omega D'')^2 f+\tfrac{1}{6}\psi^{n-3}(\chi D'+\omega D'')^3 f\ldots$$
$$+\frac{1}{1.2\ldots n}(\chi D'+\omega D'')^n f=0$$

is the equation of the contact section, where ψ, χ, ω are to be considered as the coordinates of a point in a plane; F is a symbol for the polynome on the right-hand side considered as a function of ψ, χ, ω, the coordinates of P, D, D' being treated as constant. If then the curve besides the double point P (at which point $\chi=0$, $\omega=0$) has another double point Q, then putting for shortness $\frac{dF}{d\psi}=F_\psi$, &c., the equations $F_\psi=0$, $F_\chi=0$, $F_\omega=0$ must be satisfied without χ and ω vanishing. This gives an equation between the coordinates in space of the points P, D, D'', and (as might be expected from the nature of the question) finally an equation containing only w, x, y, z, and which combined with the equation $f=0$ represents the required curve of contact of the doubly circumscribed developable. But since by reason of the double point P the resultant of the polynomes F_ψ, F_χ, F_ω vanishes *identically*, the system must be replaced by a system for which this does not happen; to effect this we may proceed as follows:

The functions F_χ, F_ω may be brought under the forms

$$F_\chi=M\chi+N\omega,\quad F_\omega=P\chi+Q\omega,$$

and the equations $F_\chi=0$, $F_\omega=0$ give therefore

$$\Delta=MQ-NP=0$$

and the function Δ for $\chi=0$, $\omega=0$ reduces itself to

$$\{(D'f)^2(D''f)^2-(D'D''f)^2\}\,\psi^{2(n-2)}.$$

* Remark. This section contains an attempt to apply Jacobi's method, given in Crelle's Journal, for the determination of the double tangents of a plane curve, to the doubly circumscribed developable of a surface.

Moreover in the development of

$$G=\frac{2}{n-2}\psi F_\psi-\chi F_\chi-\omega F_\omega=\frac{n}{n-2}\psi F_\psi-nF,$$

the lowest term in respect to χ, ω, is

$$-\frac{n}{n-2}\tfrac{1}{6}\psi^{n-3}(\chi D'+\omega D'')^3 f.$$

Considering now the resultant Θ of the system

$$F_\psi=0,\quad F_\chi=0,\quad \Delta=0,$$

this must be in the first place divisible by the resultant K of the system

$$F_\psi=0,\quad M=0,\quad N=0,$$

and in the next place by

$$\Gamma=(D'^2 f)(D''^2 f)-(D'D''f)^2,$$

since $\chi=0$, $\omega=0$, $\Gamma=0$ are also a solution of the system Θ. But since we have identically

$$\frac{2}{n-2}N\psi F_\psi=NG+(N\chi+Q\omega)F_\chi-\Delta\chi\omega,$$

and since for $\Gamma=0$ and considering χ, ω as indefinitely small quantities of the first order, the polynomes F_χ, Δ are only of the first order, but G is of the third order, Θ must be divisible by Γ^2.* As regards K there is nothing to shew that a higher power than the first enters as a factor into Θ, and a further examination shews that Θ is in fact divisible only by the first power of K.

In relation to ψ, χ, ω we have F_ψ, F_χ each of the degree $n-1$, Δ of the degree $2(n-2)$ and M, N of the degree $n-2$. The coefficient of a term $\psi^\alpha\chi^\beta\omega^\gamma$ in F_ψ is in regard to the coordinates of the points P, D, D' respectively of the degrees $\alpha+1$, β, γ, in F_χ of the degrees α, $\beta+1$, γ, in M of the degrees α, $\beta+2$, γ, in N of the degrees α, $\beta+1$, $\gamma+1$, in P the same, and in Q of the degrees

* (Note by the Translator). I do not quite understand the reasoning: but if we write $F=A\chi^2+2B\chi\omega+C\omega^2$ and take Γ the value of $AC-B^2$ corresponding to $\chi=0$, $\omega=0$, then when χ, ω are small $d_\psi A$, $d_\psi B$, $d_\psi C$ are proportional to A, B, C, and the system (Θ) may be written $A\chi^2+2B\chi\omega+C\omega^2=0$, $A\chi+B\omega=0$, $\Gamma+A_1\chi+B_1\omega=0$, the last two equations shew that (putting for shortness $AB_1-A_1B=T$) $T\chi$, $T\omega$ are respectively equal to $-B\Gamma$, $+A\Gamma$, and substituting these values in the first equation, the left-hand side of the resulting equation contains the factor $(AB^2-2BAB+CA^2)\Gamma^2$, which is equal to $A(AC-B^2)\Gamma^2$, i.e. the resultant contains the factor Γ^2.

$\alpha, \beta, \gamma+2$, conequently in Δ of the degrees $\alpha, \beta+2, \gamma+2$. Lastly, Γ is in regard to such coordinates of the degrees $2(n-2), 2, 2$. It follows that in reference to the coordinates of the three points respectively,
Θ is of the degrees

$$2n(n-1)(n-2), \quad 2(n-1)^3+2(n-1)(n-2), \quad 2(n-1)^3,$$

and K of the degrees

$$n(n-2)^2, \quad (n-1)^2(n-2)+2(n-1)(n-2), \quad (n-1)^2(n-2).$$

Whence $\dfrac{\Theta}{K\Gamma^3}$ is of the degrees

$$(n-2)(n^2-6), \quad n(n-1)^2-6, \quad n(n-1)^2-6;$$

this resultant will be denoted by Ω $\left(\Omega = \dfrac{\Theta}{K\Gamma^3}\right)$.

If we put

$$\psi = \psi' + \lambda\chi' + \mu\omega', \quad \chi = \alpha\chi' + \beta\omega', \quad \omega = \gamma\chi' + \delta\omega',$$

then in the new system of coordinates (ψ', χ', ω') the fundamental point P is the same as before, and only the two other points D, D' have assumed arbitrary new positions in the tangent plane of the basis at P. The polynome of the equation of the contact section, considered as expressed in terms of ψ', χ', ω' will have the same properties as the before mentioned polynome, it will have therefore a corresponding resultant Ω'; and since x', x'' are respectively replaced by $\lambda x + \alpha x' + \gamma x''$, $\mu x + \beta x' + \delta x''$ and similarly for the other coordinates, Ω' will be in regard to each of the series of constants λ, α, γ and μ, β, δ of the degree $n(n-1)^2-6$. But since $\psi=0$, $\chi=0$, $\omega=0$ is a solution of the new system, which implies $\alpha\delta-\beta\gamma=0$ without besides having the variable solution $\psi'=0$, $\chi'=0$, $\omega'=0$ as a necessary consequence, Ω' must be divisible by a power of $\alpha\delta-\beta\gamma$, in such manner that the quotient may differ from Ω only by a trivial constant (that is a constant independent of $\alpha, \beta, \gamma, \delta, \lambda, \mu$), we must therefore have

$$\Omega' = (\alpha\delta-\beta\gamma)^{n(n-1)^2-6}\,\Omega,$$

since for $\lambda=\mu=\beta=\gamma=0$, $\alpha=\delta=1$, Ω' and Ω must coincide. Suppose now $df = p\,dw + q\,dx + r\,dy + s\,dz$, and consequently (since the equation $f=0$ is satisfied)

$$pw + qx + ry + sz = 0,$$

whence among other relations

$$(pw+qx)(pw+ry) = qrxy - pswz,$$

And writing

$$D' = q\frac{d}{dw} - p\frac{d}{dx}, \quad D'' = s\frac{d}{dy} - r\frac{d}{dz},$$

the points D', D'' will be on the tangent plane. Putting moreover

$$\psi = \psi' + \frac{pr\chi' - qs\omega'}{pw + qx}, \quad \chi = \frac{rx\chi' + sw\omega'}{pw + qx}, \quad \omega = \frac{pz\chi' + yq\omega'}{pw + qx},$$

we have

$$\begin{aligned} & f(w\psi + q\chi, \; x\psi - p\chi, \; y\psi + s\omega, \; z\psi - r\omega) \\ & = f(w\psi' + r\chi', \; x\psi' - x\omega', \; y\psi' - p\chi', \; z\psi' + q\omega'), \end{aligned}$$

and
$$\Omega' = \left(\frac{pw + ry}{pw + qx}\right)^{n(n-1)^2-6} \Omega,$$

as before, under the supposition $f = 0$. But since as well Ω' as Ω are integral functions of $w, x, y, z : p, q, r, s$, viz. in regard to the first set of the degree $(n-2)(n^2-6)$, and in regard to the second set of the degree $2[n(n-1)^2 - 6]$, it follows that putting for p, q, r, s the values of these quantities considered as derivatives of the polynome f, we must have identically

$$(pw + qx)^{n(n-1)^2-6}\Omega' - (pw + ry)^{n(n-1)^2-6}\Omega = Vf,$$

where V is a rational and integral function of w, x, y, z. There is nothing from which it would appear that the system $f = 0$, $pw + qx = 0$, $pw + ry = 0$, or what is the same thing $pw = -qx = -ry = sz$ represents a curve and not a mere system of discrete points. But since the curve

$$pw + qx = 0, \quad pw + ry = 0$$

lies wholly in the surface $Vf = 0$, and no part of the curve lies in the surface $f = 0$, the curve must lie wholly in the surface $V = 0$, and the form of the identical equation shews that the curve in question enters as an $[n(n-1)^2 - 6]$-tuple curve of the surface $V = 0$. Now I believe that whenever a complete curve is represented by the equations $k = 0$, $l = 0$, every surface passing through the curve may be represented by an equation $kt + lu = 0$. From such an axiom it follows that, for the present case, we must have identically

$$V = (pw + qx)^{n(n-1)^2-6}T'' - (pw + ry)^{n(n-1)^2-6}T,$$

where T, T'' are rational and integral functions. And when this is once granted, it follows from known and strictly demonstrated theorems relating to the divisibility of rational functions, that we must have identically

$$\Omega = (pw + qx)^{n(n-1)^2-6} R + Tf,$$

where R is a rational and integral function.

The required curve of contact was at first contained in the system $\Theta = 0$, $f = 0$, then after the separation of extraneous curves in the system $\Omega = 0$, $f = 0$. This last system in virtue of the relation just obtained breaks up into the multiple system $pw + qx = 0$, $f = 0$, and the unique system $R = 0$, $f = 0$. The former on account of its arbitrariness cannot contain the required curve, which must therefore be contained in the latter system. But R being obviously of the degree $(n-2)(n^3 - n^2 + n - 12)$, the degree of the curve of contact is at most $n(n-2)(n^3 - n^2 + n - 12)$. We proceed to shew that the curve is actually of this degree; from which it will follow that it is a complete curve, that is, that a surface $R = 0$ passes through the curve of contact and intersects the basis only in this curve and in no other curve, if at least the axiom relied upon was not deceptive.

Imagine a cone having for its vertex a point D, circumscribed about the surface, and let it be required to find for this cone the degree g, the class k, the number of double sides d, of cuspidal (stationary) sides r, of double tangent planes t, and of stationary tangent planes w. It is clear that it is only necessary to know three of these six numbers in order to determine the others by means of the same three relations which apply to plane curves, viz.

$$n - r = 3(k - g), \quad g(g-1) = k + 2d + 3r, \quad k(k-1) = g + 2t + 3w.$$

(see Steiner's *Memoir* on the subject, Crelle, t. xlvii., and Liouville, t. xviii. p. 309; also Salmon's *Treatise on the Higher Plane Curves*, p. 91). The curve along which the cone touches the surface is defined by the system $f=0$, $Df=0$; the tangent (when Δ denotes the symbol of one of its points) by $\Delta f = 0$, $D\Delta f = 0$. Comparing this with the system $\delta f = 0$, $\delta^2 f = 0$, which determines the two tangents at the double point of the contact section; it is easy to see that the tangent $P\Delta$ of the curve of contact of the surface and circumscribed cone, and the generating line PD of the cone are harmonically related to the two tangents of the contact section at the double point.* Each generating line therefore

* This also follows easily from the more general theorem: If three surfaces touch at the same point, the pairs of tangents of the three contact sections at the point in question form a pencil in involution.

of the cone which coincides with one of the two tangents at the double point of the contact section will be also a tangent to the curve of contact of the surface with the circumscribed cone, and in particular when the point of contact of the tangent plane is a cusp of the contact section, the tangent of the curve of contact of the surface with the circumscribed cone coincides with the cuspidal tangent of the contact section, so long as the generating line of the cone has any other direction whatever. In the former case the cone has a cuspidal (stationary) generating line, in the latter a stationary tangent plane. For the cuspidal or stationary generating line the conditions are $f=0$, $Df=0$, $D^2f=0$, and we have therefore $r=n(n-1)(n-2)$. For a cusp of the contact section of the basis it is necessary that the system $\Delta f=0$, $\Delta^2 f=0$ should have in reference to the elements of Δ two coincident solutions, which may be expressed by the evanescence of ∇f (the Hessian functional determinant or Hessian). Consequently the stationary tangent planes of the cone are given by the system $f=0$, $Df=0$, $\Delta f=0$, and therefore $w=n(n-1)\times 4(n-2)$. The order g of the cone is the class of the section of the basis by a plane through the vertex of the cone, so that $g=n(n-1)$ and the class k of the cone is the class of the basis, that is, $k=n(n-1)^2$. We have already four of the required numbers, more than enough therefore to determine the two others. We find

$$d=\tfrac{1}{2}n(n-1)(n-2)(n-3),$$
$$t=\tfrac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12).$$

I stop to consider this last number t. Since this represents the number of planes passing through a given point D and touching the basis in two distinct points, it is naturally the class of the doubly circumscribed developable of the basis. But the curve of contact is intersected by the polar surface $Df=0$, obviously only in the pairs of points of contact of the planes through D; consequently the number of these points of intersection is $2t$ and the degree of the curve of contact is

$$\frac{2t}{n-1}=n(n-2)(n^3-n^2+n-12),$$

which was the number above obtained as the maximum limit of the degree of the curve. I am indebted to Dr. Steiner for this process for determining the class of the doubly circumscribed developable. The determination of the order of the circumscribed developable appears to me a very interesting problem. If it were solved, as to which I at present know nothing, we should be in a condition to derive, by means of it, the number of the triple tangent planes of the surface, and generally an explanation of all the singularities which a general (as regards order) surface presents in respect to its class.

The order in question would be determined if it could be found, how often, for example, the right line $w=0$, $x=0$ is intersected by a generating line of the developable. If we retain the symbols

$$D'=q\frac{d}{dw}-p\frac{d}{dx},\quad D''=s\frac{d}{dy}-r\frac{d}{dz},$$

the generating line in question will pass through the points P and D''. For the second double point (besides P) of the contact section we must have $\chi=0$. The former system, the resultant of which was Ω, then easily reduces itself to the following

$$\sum_{i=2}^{i=n-1}\frac{n-i}{1.2.3\ldots i}\psi^{n-i-1}\omega^{i-2}\quad D''^{i}f=0,$$

$$\sum_{i=2}^{i=n}\frac{i-2}{1.2.3\ldots i}\psi^{n-i}\ \omega^{i-2}\quad D''^{i}f=0,$$

$$\sum_{i=1}^{i=n-1}\frac{1}{1.2.3\ldots i}\psi^{n-i-1}\omega^{i-1}D'D''^{i}f=0,$$

to which is to be added $f=0$. From these four equations the four unknown quantities $\psi:\omega$, $w:x:y:z$ are to be determined and the extraneous solutions rejected. It is of course intended that p, q, r, s, which denote the first derived functions of f, should be replaced by their values. In order to give an idea how numerous the extraneous solutions may be, I may mention that for $n=3$, the system reduces itself to $f=0$, $D''f=0$, $D''^{2}f=0$, and that all the 90 solutions are extraneous, inasmuch as 18 solutions belong to the system (to be taken six times over) $w=0$, $x=0$, $f=0$, and 72 to the system (to be taken six times over) $r=0$, $s=0$, $f=0$.

In order to exhaust the singular tangent planes of a general (as regard order) surface, we must imagine the planes which touch the basis along the curve $f=0$, $\nabla f=0$, consequently in curves having a cusp at the point of contact, such planes, considered in respect to class, have two coincident points of contact, and are therefore singular tangent planes. The system of the planes in question generate what I call the osculating circumscribed developable, the curve in question may be called the osculation curve; it separates the region of the basis where the measure of curvature is negative (consequently where ∇f is positive and the two tangents at the double point of the contact section are real) from the region where the measure of curvature is positive. There are certain determinate points of the basis where the osculation curve and the curve of contact of the doubly circumscribed developable, 1° simply intersect, 2° touch. A plane which touches the basis at a point of the former kind intersects the basis in a curve having a double point and also a cusp; a plane touching the basis at a point of the latter kind cuts the basis in a curve having at the point of contact a self-touching double point, that is, a double point where the two branches touch; the tangent at such double point coincides with that of the osculation curve; and if in the neighbourhood of such a point we follow the motion of the double tangent plane, we find that upon one side of the curve of osculation the two points of contact of the plane are real points indefinitely near to each other, and on the other side the plane is still real but the two points of contact are imaginary and conjugate to each other.

With respect to these singular developables and planes I assume the numerical relations following:

1°. $a = \frac{1}{2}n(n-1)(n-2)(n^3 - n^2 + n - 12)$ the class of the doubly circumscribed developable, A the (still unknown) order.

2°. $b = 4n(n-1)(n-2)$ the class of the osculating circumscribed developable $B = 2n(n-2)(3n-4)$ its order.

3°. κ the (still unknown) number of the triple tangent planes.

4°. $\lambda = 4n(n-2)(n-3)(n^2+3n-16)$ the number of planes touching the surface in a curve having a double point and also a cusp.

5°. $\mu = 2n(n-2)(11n-24)$ the number of planes touching the surface in a curve having a self-touching double point.

The class of the surface is $k = n(n-1)^2$ If the surface were general (as regards class) the order would be $k(k-1)^2$. The difference $k(k-1)^2 - n$ is to be accounted for by means of the singular developables and tangent planes. The doubly circumscribed developable in itself (abstracting the tangent planes of a higher singularity included in it) diminishes the class of the surface by $ak + 2A$, the osculating circumscribed developable (with the like abstraction) diminishes the class by $2bk + 3B$, each triple tangent plane (abstracting the three sheets of the developable to which it is common) diminishes the class by 3, each tangent plane cutting the surface in a curve having a double point and cusp by 4, and lastly each tangent plane cutting the surface in a curve having a self-touching double point by 6. We have thus

$$(a + 2b)\, k + 2A + 3B + 3\kappa + 4\lambda + 6\mu = k\,(k - 1)^2 - n,$$

which gives between the still unknown numbers A and κ the following relation:

$$2A + 3\kappa = \tfrac{1}{2} n(n-2)(n^7 - 4n^6 + 7n^5 - 45n^4 + 118n^3 - 115n^2 + 508n - 912).$$

For $n = 3$ we have

$$a = 27, \quad b = 30, \quad B = 24, \quad \lambda = 0, \quad \mu = 54, \quad k = 12.$$

But as a curve of the third order cannot have two double points without breaking up into a conic and a right line, it is clear that the doubly circumscribed developable of a surface of the third degree can consist only of planes passing through fixed lines upon the basis, and that since the class is $a = 27$, there are upon the basis 27 such lines which play the part of the developable in question. But as these lines are not in general intersected by an arbitrary line, we must have $A = 0$ for the degree of this degenerate developable and the formula gives $\kappa = 45$ as the number of the triple tangent planes, which it is clear meet the basis in three right lines, a number which may be obtained by other considerations.

Remark by the Translator. The investigations contained in the present portion of Prof. Schläfli's Memoir, with respect to the general theory of algebraical surfaces, are similar in character to those of Mr. Salmon, and several of the author's results have been already given in Mr. Salmon's Memoirs in the *Journal*, but the theory is here carried a few steps further than in the memoirs just referred to; and the knowledge which I have of Mr. Salmon's still unpublished Memoir on Reciprocal Surfaces, in which the whole subject is considered in a more complete manner (and in particular formulæ are given leading to the determination of the two numbers A and κ) was clearly not a reason for delaying the publication of Prof. Schläfli's interesting Memoir, which was kindly sent by him for insertion in the *Journal*.

(*To be Continued.*)

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By Dr. Schläfli, Professor of Mathematics at the University of Bern.
Translated by A. Cayley.

(Continued from p. 65).

I imagine to myself a homogeneous equation of the third order in the four point coordinates w, x, y, z, where all the twenty coefficients have any values whatever. From this may be calculated the function denoted above by R, which in the present case is a function of the degree 9. The surface $R=0$ will then meet the given basis surface of the third order $f=0$, in the twenty-seven lines of this surface. If therefore the equations $f=0$, $R=0$ are combined with any two linear equations

$$l=aw+bx+cy+dz=0,\quad l'=a'x+b'y+c'z+d'w=0,$$

it must be demonstrable that the resultant of the four functions f, R, l, l' can be (in respect to the indeterminate coefficients of the linear functions l, l') decomposed into twenty-seven factors of the form

$$\begin{vmatrix} \alpha a+\beta b+\gamma c, & d \\ \alpha a'+\beta b'+\gamma c', & d' \end{vmatrix} + \begin{vmatrix} \alpha', & \beta', & \gamma' \\ a, & b, & c \\ a', & b', & c' \end{vmatrix}$$

where the constants $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ satisfy the condition $\alpha\alpha'+\beta\beta'+\gamma\gamma'=0$. And then there will pass through the line corresponding to any such factor, the four planes

$$\gamma x-\beta y+\alpha' z=0,$$
$$-\gamma w+\alpha y+\beta' z=0,$$
$$\beta w-\alpha x+\gamma' z=0,$$
$$-\alpha' w-\beta' x-\gamma' y=0.$$

Suppose that one line of the given basis surface $f=0$ is known: and let the system of coordinates be transformed in such manner that two fundamental planes s, t, pass through the line in question. The equation of the surface will not contain any part not divisible by s or t, and it can therefore be reduced to the form $\begin{vmatrix} s, & S \\ t, & T \end{vmatrix} = 0$, where S and T denote polynomes of the second order. The basis surface contains therefore all the conics represented by the two equations $s + \lambda t = 0$, $S + \lambda T = 0$, where λ is an arbitrary constant. But λ can be so disposed of that the conic may break up into a pair of lines: the condition for this is, in regard to λ, of the fifth order; consequently, through each line of the basis there pass five planes, each of which intersects the basis in the three sides of a triangle, and the number of such planes is $\frac{27.5}{3} = 45$. Suppose that λ, μ are two different constants, satisfying the condition in question: the equation of the basis can then be brought under the form $\begin{vmatrix} s+\lambda t, & S+\lambda T \\ s+\mu t, & S+\mu T \end{vmatrix} = 0$, which may be denoted more simply by $\begin{vmatrix} u, & U \\ x, & X \end{vmatrix} = 0$. Here U, X are polynomes belonging to surfaces of the second order, which are respectively touched by the planes u, x. If p is the polynome of any other plane which touches both of the surfaces U, X, then there exists a constant α for which $U + \alpha pu$ breaks up into two factors, and in like manner a constant β for which $X + \beta pu$ breaks up into two factors. The plane p belongs to a developable of the fourth class, and has as such a single motion, i.e. its equation contains a single arbitrary parameter. We may therefore impose another condition, and write $\alpha = \beta$. Replace αp by the single letter p, and take D, Δ as symbols of the points in which the surfaces U, X are touched by the planes u, x respectively. Since then, each of the polynomes $U + pu$, $X + px$ breaks up into factors, it is clear that the equations

$$D(U + pu) = 0, \ \Delta(X + px) = 0$$

will be satisfied identically. But obviously, $DU = au$, $\Delta X = bx$, where a and b are constants, and $Du = 0$, $\Delta x = 0$. The foregoing equations become therefore $a + Dp = 0$, $b + \Delta p = 0$, whence $\begin{vmatrix} a, & Dp \\ b, & \Delta p \end{vmatrix} = 0$, or if we please $\begin{vmatrix} Du, & uDp \\ \Delta x, & x\Delta p \end{vmatrix} = 0$, (the left-hand side divisible by ux) an equation which is homogeneous and linear in respect to the coefficients of p; that is, there exists a fixed point through which the simply moveable plane p always passes. The problem has therefore four solutions. And if we select at pleasure one of the four polynomes p which satisfy the required conditions, and write $U+pu=-yz$, $X+px=vw$, the equation of the basis becomes

$$\begin{vmatrix} u, & -yz \\ x, & vw \end{vmatrix} = uvw + xyz = 0.$$

The possibility of such a transformation might have been seen *à priori*, since the six linear polynomes u, &c., contain 18 ratios of coefficients, to which is to be added a constant factor contained in one of the products xyz, uvw; so that there are in all 19 disposable constants, which is precisely the number of conditions to be satisfied. We may call uvw a trihedral, and say that in the equation $uvw+xyz=0$, the basis is referred to a pair of trihedrals.

Six linear polynomes are connected together by two independent linear homogeneous equations. We may multiply one of these by an arbitrary factor, and add it to the second, and the relation so obtained will of course be satisfied. Let such a relation be

$$Au+Bv+Cw+Dx+Ey+Fz=0,$$

where the coefficients are considered as containing a single arbitrary multiplier. It follows then, that

$$Au(Bv+Dx)(Cw+Dx)+Dx(Au+Ey)(Au+Fz)$$
$$=ABCuvw+DEFxyz,$$

consequently, that if $ABC=DEF$, the function on the left-hand side is a new expression for the polynome of the basis. The equation $ABC=DEF$ is, in regard to the arbitrary constant contained implicitly in the coefficients, of the degree 3, and gives therefore 3 solutions, which may be thus represented,

$$au+bv+cw+dx+ey+fz=0, \quad abc=def,$$
$$a'u+b'v+c'w+d'x+e'y+f'z=0, \quad a'b'c'=d'e'f',$$
$$a''u+b''v+c''w+d''x+e''y+f''z=0, \quad a''b''c''=d''e''f'',$$

there are thus in all 27 such transformations into trihedral pairs such as

$$au(bv+dx)(cw+dx)+dx(au+ey)(au+fz)=0.$$

The original trihedral pair $uvw+xyz=0$, gives immediately nine lines. We may for shortness represent the line $(u=0, x=0)$ by $\overline{ux}$. We have besides, 18 systems such as $(au+dx=0,\ bv+ey=0,\ cw+fz=0)$, where the third equation is always a consequence of the two others; these systems represent the other 18 lines, which may be comprised in the following two schemes,

through	$\overline{ux}, \overline{vy}, \overline{wz}$	pass	l, l', l''
"	$\overline{uy}, \overline{vz}, \overline{wx}$	"	m, m', m''
"	$\overline{uz}, \overline{vx}, \overline{wy}$	"	n, n', n''

through	$\overline{ux}, \overline{vz}, \overline{wy}$	pass	p, p', p''
"	$\overline{uz}, \overline{vy}, \overline{wx}$	"	q, q', q''
"	$\overline{uy}, \overline{vx}, \overline{wz}$	"	r, r', r''

Two lines which belong to one and the same scheme do not intersect, when they are either lines represented by the same letter differently accented, or by different letters similarly accented; but they intersect when letters and accents are both different. And two lines belonging to different schemes, intersect when the accents are the same, and do not intersect when the accents are different.

Of the 45 triangle planes, 6 form the original trihedral pair. 27 more are represented by equations such as $au+dx=0$. We represent the plane $au+dx=0$, by $(\overline{ux})$, the plane $a'u+d'x=0$, by $(\overline{ux})'$, and so in similar cases. The following scheme shews the three lines contained in each plane.

$\overline{ux}, l, p$	$\overline{vx}, n, r$	$\overline{wx}, m, q$
$\overline{uy}, m, r$	$\overline{vy}, l, q$	$\overline{wy}, n, p$
$\overline{uz}, n, q$	$\overline{vz}, m, p$	$\overline{wz}, l, r$

and similarly with one or two accents. Finally the 12 remaining planes are 6 planes such as $lm'n''$, and 6 planes such as $pq'r''$, in the representation of which the accents may be omitted since the permutation of the letters is alone sufficient. The last mentioned planes admit of no very symmetrical representation. The plane (lmn) for example has among other forms of its equation the following,

$$\frac{cd'-c'd}{c'd}(au+dx)-\frac{bf'-b'f}{bf'}(bv+ey)=0.$$

Any two triangle planes which have no line in common, determine a third plane which forms with them a trihedral, and this again determines the other trihedral of the pair. There are thus in all $\frac{45.32}{6.2}=120$ trihedral pairs, that is, the problem to reduce the equation of the basis to the form $uvw+xyz=0$, is of the degree 120. Each trihedral pair gives immediately only nine lines. It is always possible to place together three trihedral pairs to give all the twenty-seven lines; and one pair determines by itself the other two pairs. There are thus in all 40 such triads of trihedral pairs, the following is a scheme of such triads,

1 triad

$$uvw+xyz,$$
$$(lmn)(mnl)(nlm)+(lnm)(nml)(mln),$$
$$(pqr)(qrp)(rpq)+(prq)(rqp)(qpr),$$

27 triads such as

$$u(vx)(wx)+x(uy)(uz),$$
$$(vy)'(wz)''(prq)+(vy)''(wz)'(pqr),$$
$$(vz)'(wy)''(lnm)+(vz)''(wy)'(lmn),$$

12 triads such as

$$u(lmn)(prq)+(ux)(uy)'(uz)'',$$
$$v(nlm)(rqp)+(vx)(vy)'(vz)'',$$
$$w(mnl)(qpr)+(wx)(wy)'(wz)''.$$

Choosing from each pair of any triad a single trihedral, we obtain nine planes which intersect the basis in all the 27 lines. The polynome of the ninth degree above represented by R, can therefore in 320 ways be combined with the polynome f of the basis, so as to break up into linear factors. An easier survey of the 27 lines of the basis f may be arrived at as follows. We have

$$2(uvw+xyz)=\begin{vmatrix} 0 & u & x \\ y & 0 & v \\ w & z & 0 \end{vmatrix}=0,$$

this equation by linear combinations of the lines and columns may be exhibited in the more general form

$$\begin{vmatrix} r, & s, & t \\ r', & s', & t' \\ r'', & s'', & t'' \end{vmatrix}=0,$$

where all the elements of the determinant are linear functions of u, v, w, x, y, z. Hence every point determined by a system of equations such as

$$p=\alpha r+\beta s+\gamma t=0,\quad p'=\alpha' r+\beta' s+\gamma' t=0,\quad p''=\alpha'' r+\beta'' s+\gamma'' t=0,$$

will lie on the basis, and conversely the ratios $\alpha:\beta:\gamma$ may be determined for a given point of the basis. But if the condition is imposed that the polynomes p, p', p'' shall be connected by an identical equation, such as $\kappa p+\kappa' p'+\kappa'' p''=0$, in other words, that the three planes shall intersect not in a point but in a line, we arrive at the condition that all the determinants of a rectangular matrix with three horizontal and three vertical lines, the elements of which are all linear homogeneous functions of α, β, γ, vanish. It is then clear that this problem has six solutions. If we assume for example that $\kappa p+\kappa' p'+\kappa'' p''=0$ is an identical equation, the equation of the basis may be exhibited in the form

$$\begin{vmatrix} 0, & \kappa s+\kappa' s'+\kappa'' s'', & \kappa t+\kappa' t'+\kappa'' t'' \\ p', & s', & t' \\ p'', & s'', & t'' \end{vmatrix}=0,$$

which shows that each line $(p=0,\ p'=0,\ p''=0)$ corresponds to a line $(\Sigma\kappa r=0,\ \Sigma\kappa s=0,\ \Sigma\kappa t=0)$ which it does *not* intersect. But if α, β, γ belong to a different solution, and the corresponding polynomes are denoted by q, q', q'', then we have

$$\begin{vmatrix} \Sigma\kappa q, & \Sigma\kappa s, & \Sigma\kappa t \\ q', & s', & t' \\ q'', & s'', & t'' \end{vmatrix}=0$$

for the equation of the basis, and it is clear that now the two lines $(\Sigma\kappa q=0,\ q'=0)$ and $(\Sigma\kappa q=0,\ \Sigma\kappa s=0)$ intersect, since the systems have in common the equation $\Sigma\kappa q=0$. Each of the six lines represented by a system such as $(p=0,\ p'=0,\ p''=0)$ intersects all the five non-corresponding lines given by a system such as $(\Sigma\kappa r=0,\ \Sigma\kappa s=0,\ \Sigma\kappa t=0)$, and only the two corresponding lines do not intersect. I call such group of 12 lines of the basis a "double-six." It is clear that no two lines belonging to the same six intersect. The number of all the double-sixes is 36. For since each line is intersected by 10 other lines, there remain 16 lines by which it is not intersected. There are therefore $\frac{27\cdot16}{2}=216$ pairs of lines which do not intersect. Through one of the lines of such a pair pass five lines which do not intersect the other line of the pair; this other line and the five lines form together a six, and these completely determine the other six of the double-six. But of such pairs of corresponding lines as the first-mentioned pair there are in the double-six only 6; consequently $\frac{216}{6} = 36$ is the number of the double-sixes.

If now we start from the equation

$$\begin{vmatrix} 0, & u, & x \\ y, & 0, & v \\ w, & z, & 0 \end{vmatrix} = 0,$$

we have at once three solutions of the problem, to make the polynomes $\beta u + \gamma x$, $\alpha y + \gamma v$, $\alpha w + \beta z$ dependent on each other, namely $(\beta = 0,\ \gamma = 0)$, $(\alpha = 0,\ \gamma = 0)$, $(\alpha = 0,\ \beta = 0)$; the other three are obtained as follows: Suppose that

$$\kappa(\beta u + \gamma x) + \kappa'(\alpha y + \gamma v) + \kappa''(\alpha w + \beta z) = 0$$

is the identical relation between the three polynomes, and

$$Au + Bv + Cw + Dx + Ey + Fz = 0$$

the general identical relation, where A, &c. are to be considered as linear functions of a single disposable quantity. We must therefore write

$$A = \kappa\beta, \quad B = \kappa'\gamma, \quad C = \kappa''\alpha, \quad D = \kappa\gamma, \quad E = \kappa'\alpha, \quad F = \kappa''\beta,$$

which give $ABC = DEF$. This equation admits, as we know already, of three solutions. And preserving the former notations, we thus arrive at the double-six

$$\begin{pmatrix} \overline{uz}, & \overline{vx}, & \overline{wy}, & l, & l', & l'' \\ \overline{vy}, & \overline{wz}, & \overline{ux}, & n, & n', & n'' \end{pmatrix},$$

where no two lines of the same horizontal row and no two lines of the same vertical row intersect, but any two lines otherwise selected do intersect.

By means of the double-sixes we arrive, as already noticed, at an easy survey of the 27 lines and 45 planes of the basis. For represent a double-six by

$$\begin{pmatrix} a_1, & a_2, & a_3, & a_4, & a_5, & a_6 \\ b_1, & b_2, & b_3, & b_4, & b_5, & b_6 \end{pmatrix},$$

the two intersecting lines a_1, b_2 belong to a triangle which I represent by 12 and its third side by c_{12}. This third side c_{12} forms with a_2, b_1 a triangle which I represent by 21. We have thus fifteen lines c, each of which intersects only those four lines a, b, the suffixes of which belong to the pair of numbers forming the suffix of the c. And any two c's, the suffixes of which have a number in common, do not intersect; but two c's, the suffixes of which have no number in common, do intersect. There are consequently triangles such as c_{12}, c_{34}, c_{56} which may be represented by (12, 34, 56), where as well the numbers *inter se* of each pair, as the three pairs *inter se*, may be permuted. We have therefore 30 triangles such as 12, and 15 triangles such as (12, 34, 56), in all 45. Finally there are 10 trihedral pairs such as

(12) (23) (31) + (13) (32) (21)

(45) (56) (64) + (46) (65) (54)

(14,25,36)(15,26,34)(16,24,35)+(14,26,35)(16,25,34)(15,24,36)

and 30 trihedral pairs such as

(35) (46) (12, 36, 45) + (36) (45) (12, 35, 46)

(51) (62) (16, 25, 34) + (52) (61) (15, 26, 34)

(13) (24) (14, 23, 56) + (14) (23) (13, 24, 56)

The double-sixes give rise to the remark that there is here exposed to view an apparently very elementary theorem which may be thus enuntiated: "Drawing at pleasure five lines a, b, c, d, e which meet a line F, then may any four of the five lines be intersected by another line besides F. Suppose that A, B, C, D, E are the other lines intersecting (b, c, d, e), (c, d, e, a), (d, e, a, b), (e, a, b, c), and (a, b, c, d) respectively. Then A, B, C, D are intersected by the line e; there must be another line f intersecting these four lines, and this line will of itself intersect the remaining line E; i.e. there will be a line f intersecting the five lines A, B, C, D, E." Is there, for this elementary theorem, a demonstration more simple than the one derived from the theory of cubic forms?

When the equation of the cubic surface referred to a real system of coordinate axes, has all its coefficients real, it is easy to see that the surface will be real. The question however arises, how many of the 27 lines and 45 planes may be imaginary? The complete investigation would be tedious, and I content myself in giving a mere survey of the species into which the general surface of the third order divides itself in regard to the reality of the 27 lines. There are only the five species following:

A. All the 27 lines and 45 planes are real.

B. 15 lines and 15 planes are real. The twelve imaginary lines form a double-six, where each line of the one six is conjugate to the corresponding and therefore not intersecting line of the other six, wherefore none of the imaginary lines have a real point. Any two pairs of corresponding imaginary lines are intersected by a real line; and as many ways as the double-six can be divided into thrice two such pairs, in so many ways do the corresponding real lines form a triangle, viz. there are fifteen real triangles.

C. 7 lines and 5 planes are real: namely, through a real line there pass 5 real planes, but of these three only contain real triangles, in each of the other two the triangle consists of the original real line and two imaginary lines meeting in a real point.

D. 3 lines and 13 planes are real: namely, there is one real triangle, and through each side there pass (besides the plane of the triangle) 4 real planes.

E. 3 lines and 7 planes are real: namely, there is a real triangle, and through each side there pass (besides the plane of the triangle) 2 real planes.

With respect to the reality or non-reality of the six linear polynomes in the expression $uvw + xyz$, which is equivalent to a given cubic polynome with real coefficients, the following 13 cases have to be distinguished. I call them forms of the trihedral pair $uvw + xyz = 0$, and I shew in the following enumeration to which species of cubic surface each form belongs: instead of linear polynome the word plane may be used.

1°. All the six planes of the trihedral pair are real. This form occurs only in the species *A* and *B*.

2°. u and x, v and y, w and z are conjugate to each other; that is, the two trihedrals of the pair are imaginary and conjugate. In *B* and *C*.

3°. u, v, w, x are real, y and z conjugate. In *D* and *E*.

4°. u and x are real, v and w, and y and z conjugate to each other. In *C* and *E*.

5°. u and x are real, the four others imaginary, but no two of them conjugate: but v and w have their real line in x, and y and z their real line in u. (Every imaginary plane contains of course a real line). In *B* and *C*.

6°. u and x are real, the four others imaginary and no two of them conjugate: and u alone intersects y, z in real lines. In *C* and *E*.

7°. u and x are real, the four others imaginary and not conjugate. Neither u nor x have a real triangle. In *D* and *E*.

8°. u and x are conjugate, the four others are imaginary and not conjugate; v and y have a real point in common, and so have w and z. In *C* and *E*.

9°. u is real, the five other planes are imaginary and not conjugate, u intersects x in a real line, and y, z in conjugate lines. And y alone has with each of the planes v and w, a real point in common. In E.

10°. All the six planes are imaginary and not conjugate; u and x have in common a real point, v and y a real line, and w and z a real line. In C.

11°. All the six planes are imaginary and not conjugate, each plane of the one trihedral has in common with each plane of the other trihedral a real point. In D.

12°. All the six planes are imaginary and not conjugate; u has in common with x a real point, and also with y, and also with z; moreover x has a real point in common with v, and also with w.

13°. All the six planes are imaginary and not conjugate; u has a real point in common with x, and so also v with y, and w with z. In E.

If in any one of these thirteen forms the particular complete character of each of the six linear polynomes is represented explicitly, and then the transformation is undertaken of this form into another trihedral pair, it often happens that a root of the cubic equation which has to be solved for this purpose can be rationally represented by the constants of the form without the necessity of extracting a cube root. Two trihedral pair forms thus easily transformable the one into the other may be termed *equivalent;* when the one of them presents itself in any two species of the surface, the other also presents itself in the same two species. It is only the two other roots of the above mentioned cubic equation $ABC = DEF$ which decide, according as they happen to be real or imaginary, to which of the two species the surface belongs, and they give rise to a transformation complicated with a square root; trihedral pairs thus transformed into each other, on account of the possible transition from one species into a different one, I call *non-equivalent;* the more so that the discussion of the one form does not render unnecessary that of the other. In this sense

The forms 2°, 5° are equivalent and occur in B and C,
" 4°, 6°, 8° " " C and E,
" 3°, 7° " " D and E,
" 9°, 12°, 13° " " E,

while, on the contrary, the following forms are each of them isolated, viz. 1° in A and B, 10° in C, and 11° in D.

The forms of the triads of trihedral pairs arrange themselves as follows:

A has 40 triads (1, 1, 1).

B has 10 triads (1, 2, 2) and 30 triads (5, 5, 5).

C has 4 triads (2, 2, 4), 12 triads (5, 8, 8), and 24 triads (6, 10, 10).

D has 16 triads (3, 11, 11) and 24 triads (7, 7, 7).

E has 2 triads (4, 4, 4), 4 triads (3, 13, 13), 6 triads (7, 8, 8), 12 triads (6, 12, 12), and 16 triads (9, 9, 9).

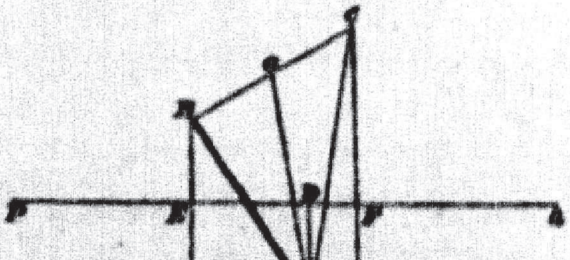
In conclusion I remark that the double-sixes play a part in the theory of the nodes of a cubic surface. I call "node" any point (w, x, y, z) of an algebraical surface $f(w, x, y, z) = 0$, for which $Df = 0$ is satisfied for all values of the four elements of the differentiation system D, and "proper node" a point at which the cone of the second order represented by $D^2f = 0$ does not break up into a pair of planes. If a surface of the 3rd order $f = 0$ has a proper node (w, x, y, z), then the six lines passing through such node and represented by the equations $D^2f = 0$, $D^3f = 0$ form a double-six, in which each two corresponding (non-intersecting) lines of the two sixes coincide.

NOTE ON THE RELATION BETWEEN THE DISTANCES OF A STRAIGHT LINE FROM THREE GIVEN POINTS.

By N. M. Ferrers, Gonville and Caius College, Cambridge.

In Salmon's *Higher Plane Curves*, Art. IX., p. 10, an investigation is given of the relation existing between the tangential coordinates of a straight line, that is, between its distances from three arbitrarily chosen points.

The result is there arrived at through the intervention of Cartesian coordinates, but it appears desirable to place the system of tangential coordinates on an independent basis, and therefore to supply a proof of the relation in question not depending on any other system.



Let A, B, C be the three points of reference, PQ any straight line, draw AD, BE, CF, severally perpendicular to PQ. Then, in accordance with the convention with respect to the use of the negative sign, if AD be considered positive, BE and CF, lying wholly *without* the angles ABC, ACB respectively, will be negative. Let then $AD = \alpha$, $BE = -\beta$, $CF = -\gamma$. Through A draw $E'F'$ parallel to PQ, produce BE, CF to meet $E'F'$ in E', F' respectively, then

$$BE' = -(\beta + \alpha), \quad CF' = -(\gamma + \alpha).$$

Bisect the angle BAC by the straight line AO, and let $OAD = \theta$.

Let $$BC = a, \quad CA = b, \quad AB = c.$$

Then $$-\frac{\beta + \alpha}{c} = \frac{BE'}{BA} = \sin BAE' = \cos BAD = \cos\left(\frac{A}{2} + \theta\right).$$

Similarly $$-\frac{\gamma + \alpha}{b} = \cos\left(\frac{A}{2} - \theta\right),$$

therefore $$-\frac{1}{2\cos\frac{A}{2}}\left(\frac{\beta + \alpha}{c} + \frac{\gamma + \alpha}{b}\right) = \cos\theta,$$

$$\frac{1}{2\sin\frac{A}{2}}\left(\frac{\beta + \alpha}{c} - \frac{\gamma + \alpha}{b}\right) = \sin\theta;$$

therefore, adding squares and simplifying,

$$\frac{1}{\sin^2 A}\left\{\frac{(\beta + \alpha)^2}{c^2} + \frac{(\gamma + \alpha)^2}{b^2}\right\} - \frac{2\cos A}{\sin^2 A}\frac{(\beta + \alpha)(\gamma + \alpha)}{bc} = 1;$$

therefore

$$b^2(\beta + \alpha)^2 + c^2(\gamma + \alpha)^2 - 2bc\cos A(\beta + \alpha)(\gamma + \alpha) = b^2c^2\sin^2 A,$$

or $$b^2(\beta + \alpha)^2 + c^2(\gamma + \alpha)^2 - (b^2 + c^2 - a^2)(\beta + \alpha)(\gamma + \alpha) = 4K^2,$$

K denoting the area of the triangle ABC; a result which may be put into the symmetrical form,

$$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 - (b^2 + c^2 - a^2)\beta\gamma - (c^2 + a^2 - b^2)\gamma\alpha - (a^2 + b^2 - c^2)\alpha\beta = 4K^2,$$

which is substantially identical with that given by Mr. Salmon, in the article above referred to.
